Exercises from Atiyah-MacDonald
Introduction to Commutative Algebra

Joshua Ruiter

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Exercises from *Atiyah-MacDonald*:

1. (Chapter 4, page 55) 4.5, 4.10
2. (Chapter 6, page 78) 6.1, 6.5, 6.7, 6.8
3. (Chapter 8, page 92) 8.2, 8.3
4. (Chapter 10, page 113) 10.4, 10.9, 10.10
5. (Chapter 11, page 125) 11.1, 11.4

1 Chapter 4

Proposition 1.1 (Exercise 4.5). *Let K be a field, and let $A = K[x, y, z]$. Consider the ideals*

$$\mathfrak{p}_1 = (x, y) \qquad \mathfrak{p}_2 = (x, z) \qquad \mathfrak{m} = (x, y, z)$$

Note that $\mathfrak{p}_1, \mathfrak{p}_2$ are prime, and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Then $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Consequently, the associated primes of $\mathfrak{a}$ are $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{m}$. Of these, $\mathfrak{p}_1, \mathfrak{p}_2$ are isolated, and $\mathfrak{m}$ is embedded.

Proof. It is not too hard to see that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Clearly $\mathfrak{p}_1, \mathfrak{p}_2$ are primary because they are prime, and by Proposition 4.2 of Atiyah-MacDonald, $\mathfrak{m}^2$ is primary. To show that it is reduced, we need to show the following three containments fail.

$$\mathfrak{p}_1 \cap \mathfrak{p}_2 \not\subset \mathfrak{m}^2 \qquad \mathfrak{p}_1 \cap \mathfrak{m}^2 \not\subset \mathfrak{p}_2 \qquad \mathfrak{p}_2 \cap \mathfrak{m}^2 \not\subset \mathfrak{p}_1$$

In each case, we just need a single element.

$$x \in (\mathfrak{p}_1 \cap \mathfrak{p}_2) \setminus \mathfrak{m}^2 \qquad y^2 \in (\mathfrak{p}_1 \cap \mathfrak{m}^2) \setminus \mathfrak{p}_2 \qquad z^2 \in (\mathfrak{p}_2 \cap \mathfrak{m}^2) \setminus \mathfrak{p}_1$$

Thus $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. The associated primes are the radicals of the primes appearing in the decomposition. For $\mathfrak{p}_2, \mathfrak{p}_2$, they are equal to their own radical, but the radical of $\mathfrak{m}^2$ is $\mathfrak{m}$, so $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{m}$ are the associated primes of $\mathfrak{a}$. Clearly, $\mathfrak{p}_1, \mathfrak{p}_2$ are minimal, so they are the isolated primes, and $\mathfrak{m}$ is not, so it is embedded. □

Proposition 1.2 (Exercise 4.10)**.** *Let $\mathfrak{p}$ be a prime ideal in a ring A, and let $A \to A_\mathfrak{p}, a \mapsto \frac{a}{1}$ be the canonical homomorphism, and let $S_\mathfrak{p}(0)$ be the kernel. Then*

1. $S_\mathfrak{p}(0) \subset \sqrt{S_\mathfrak{p}(0)} \subset \mathfrak{p}$
2. *If $\mathfrak{q} \subset A$ is another prime ideal such that $\mathfrak{p} \supset \mathfrak{q}$, then $S_\mathfrak{p}(0) \subset S_\mathfrak{q}(0)$.*
3. $\sqrt{S_\mathfrak{p}(0)} = \mathfrak{p}$ *if and only if $\mathfrak{p}$ is a minimal prime of A.*
4. *Let $D(A)$ be the prime ideals of A such that there exists $a \in A$ with $\mathfrak{p}$ minimal among primes containing* $\mathrm{Ann}(a)$*. Then*
$$\bigcap_{\mathfrak{p} \in D(A)} S_\mathfrak{p}(0) = 0$$

(See Exercise 4.8 of Atiyah-MacDonald for other properties of $D(A)$.)

Proof. (1) The inclusion $I \subset \sqrt{I}$ is true for any ideal. For the second inclusion, we start by observing that

$$S_\mathfrak{p}(0) = \left\{ a \in A : \frac{a}{1} = \frac{0}{1} \text{ in } A_\mathfrak{p} \right\} = \{a \in A : \exists s \in A \setminus \mathfrak{p} \text{ such that } sa = 0 \text{ in } A\} = \bigcup_{s \in A \setminus \mathfrak{p}} \mathrm{Ann}(s)$$

$$\sqrt{S_\mathfrak{p}(0)} = \{x \in A : \exists s \in A \setminus \mathfrak{p} \text{ and } n > 0 \text{ such that } sx^n = 0\} = \bigcup_{s \in A \setminus \mathfrak{p}} \sqrt{\mathrm{Ann}(s)}$$

If $x \in \sqrt{S_\mathfrak{p}(0)}$, we have $sx^n = 0 \in \mathfrak{p}$ with $s \in A \setminus \mathfrak{p}$, and since $\mathfrak{p}$ is prime, $x^n \in \mathfrak{p}$. Then again by primality (and a mild induction), $x \in \mathfrak{p}$.

(2) Suppose $\mathfrak{q} \subset \mathfrak{p}$. Let $a \in S_\mathfrak{p}(0)$, so there exists $s \in A \setminus \mathfrak{p}$ with $sa = 0$. Since $\mathfrak{q} \subset \mathfrak{p}$, $A \setminus \mathfrak{p} \subset A \setminus \mathfrak{q}$, so $s \in A \setminus \mathfrak{q}$, so $a \in S_\mathfrak{q}(0)$. Thus $S_\mathfrak{p}(0) \subset S_\mathfrak{q}(0)$.

(3) Suppose $\sqrt{S_\mathfrak{p}(0)} = \mathfrak{p}$. We want to show that $\mathfrak{p}$ is a minimal prime, so suppose $\mathfrak{q} \subset \mathfrak{p}$ for some prime $\mathfrak{q}$. By (2), $S_\mathfrak{p}(0) \subset S_\mathfrak{q}(0)$, so $\sqrt{S_\mathfrak{p}(0)} \subset \sqrt{S_q(0)}$. Putting this together with (1), we obtain

$$\mathfrak{p} = \sqrt{S_\mathfrak{p}(0)} \subset \sqrt{S_\mathfrak{q}(0)} \subset \mathfrak{q}$$

Hence $\mathfrak{p} = \mathfrak{q}$, so $\mathfrak{p}$ is minimal. Conversely, suppose $\mathfrak{p}$ is a minimal prime. By the ideal correspondence with $A_\mathfrak{p}$, this is equivalent to saying that $A_\mathfrak{p}$ has a unique prime, namely $\mathfrak{p}A_\mathfrak{p}$. Thus the nilradical of $A_\mathfrak{p}$ is precisely $\mathfrak{p}A_\mathfrak{p}$. That is to say, for $x \in \mathfrak{p}$, there exists n so that $\left(\frac{x}{1}\right)^n = 0$ in $A_\mathfrak{p}$, which is to say that $x^n \in S_\mathfrak{p}(0)$. Thus $x \in \sqrt{S_\mathfrak{p}(0)}$. We have shown that $\mathfrak{p} \subset \sqrt{S_\mathfrak{p}(0)}$, and the reverse inclusion is shown in (1), so we get the desired equality.

(4) Let $x \in A, x \neq 0$. Then choose a prime $\mathfrak{p}$ which is minimal among primes containing $\mathrm{Ann}(x)$. Then

$$S_\mathfrak{p}(0) = \{a \in A : \exists s \in A \setminus \mathfrak{p} : sa = 0\}$$

Since $\mathrm{Ann}(x) \subset \mathfrak{p}$, $\mathrm{Ann}(x) \cap A \setminus \mathfrak{p} = \emptyset$. That is, $x \notin S_\mathfrak{p}(0)$. Thus

$$x \notin \bigcap_{\mathfrak{p} \in D(a)} S_\mathfrak{p}(0)$$

so the intersection contains no nonzero elements. □

2 Chapter 6

Lemma 2.1 (for Exercise 6.1)**.** *Let A be a ring, let M be an A-module, and let $\phi \in \mathrm{End}_A(M)$.*

1. *Suppose ϕ is surjective. Then ϕ is injective if and only if $\ker \phi^n = \ker \phi^{n+1}$ for some n.*

2. *Suppose ϕ is injective. Then ϕ is surjective if and only if $\operatorname{coker} \phi^n = \operatorname{coker} \phi^{n+1}$ for some n.*

Proof. (1) The forward implication is obvious and does not even require the surjectivity hypothesis. For the converse, consider the following commutative diagram with exact rows, with $n \geq 1$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \phi^n & \hookrightarrow & M & \overset{\phi^n}{\twoheadrightarrow} & M & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \iota_n} & & \big\downarrow{\scriptstyle \mathrm{Id}_M} & & \big\downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \ker \phi^{n+1} & \hookrightarrow & M & \underset{\phi^{n+1}}{\twoheadrightarrow} & M & \longrightarrow & 0 \end{array}$$

By the Snake Lemma, there is an exact sequence

$$0 = \ker \mathrm{Id}_M \to \ker \phi \to \operatorname{coker} \iota_n \to \operatorname{coker} \mathrm{Id}_M = 0$$

Thus $\ker \phi \cong \operatorname{coker} \iota_n$. If $\ker \phi^n = \ker \phi^{n+1}$ for some n, then ι_n is surjective for some n, so it has trivial cokernel, so $\ker \phi = 0$.

(2) The forward implication is obvious and does not even require the injectivity hypothesis. For the converse, consider the following commutatie diagram with exact rows, with $n \geq 1$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\phi^{n+1}} & M & \twoheadrightarrow & \operatorname{coker} \phi^{n+1} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \mathrm{Id}_M} & & \big\downarrow{\scriptstyle \pi_n} & & \\ 0 & \longrightarrow & M & \xrightarrow{\phi^{n}} & M & \twoheadrightarrow & \operatorname{coker} \phi^{n} & \longrightarrow & 0 \end{array}$$

where π_n is the map $\overline{x} \mapsto \overline{x}$ (one checks quickly that this is well-defined). By the Snake Lemma, there is an exact sequence

$$0 = \ker \mathrm{Id}_M \to \ker \pi_n \to \operatorname{coker} \phi \to \operatorname{coker} \mathrm{Id}_M = 0$$

thus $\ker \pi_n \cong \operatorname{coker} \phi$. If $\operatorname{coker} \phi^{n+1} = \operatorname{coker} \phi^n$ for some n, then π_n is injective for some n, so it has trivial kernel, so $\operatorname{coker} \phi = 0$. □

Proposition 2.2 (Exercise 6.1)**.** *Let A be a ring, let M be an A-module, and let $\phi \in \mathrm{End}_A(M)$.*

1. *If M is a Noetherian A-module and ϕ is surjective, then ϕ is also injective.*

2. *If M is an Artinian A-module and ϕ is injective, then ϕ is also surjective.*

Proof. (1) Consider the chain of A-submodules of M,

$$0 = \ker \phi^0 \subset \ker \phi^1 \subset \ker \phi^2 \subset \ker \phi^3 \subset \cdots$$

Since M is Noetherian, this stabilizes and $\ker \phi^n = \ker \phi^{n+1}$ for some n. Then by part (1) of Lemma 2.1, ϕ is injective.

(2) Consider the chain of A-submodules of M,

$$\operatorname{coker} \phi \supset \operatorname{coker} \phi^2 \supset \operatorname{coker} \phi^3 \supset \cdots$$

Since M is Artinian, this stabilizes and $\operatorname{coker} \phi^n = \operatorname{coker} \phi^{n+1}$ for some n. Then by part (2) of Lemma 2.1, ϕ is surjective. □

Definition 2.1. A topological space X is **Noetherian** if the open subsets of X satisfy the ascending chain condition. That is, if we have open subsets of X,

$$U_1 \subset U_2 \subset U_3 \subset \cdots$$

then eventually this stabilizes, $U_n = U_{n+1} = \cdots$. Equivalently, the closed subsets of X satisfy the descending chain condition.

Proposition 2.3 (Exercise 6.5)**.** *Let X be a Noetherian topological space. Then*

1. *Every subspace of X is Noetherian.*
2. *X is quasi-compact (every open cover has a finite subcover).*
3. *Every subspace of X is quasi-compact.*

Proof. (1) Let $A \subset X$ be a subset, endowed with the subspace topology, and let

$$U_1 \subset U_2 \subset U_3 \subset \cdots$$

be an ascending chain of open subsets of A. By definition of the subspace topology, $U_i = A \cap V_i$ for some open subsets $V_i \subset X$. Define

$$V_n' = \bigcup_{i=1}^{n} V_i$$

Then

$$V_1' \subset V_2' \subset V_3' \subset \cdots$$

is an ascending chain of open subsets of X, so by the Noetherian property it stabilizes, so for some n, we have

$$\bigcup_{i=1}^{n} V_i = \bigcup_{i=1}^{n+1} V_i \qquad \text{equivalently,} \qquad V_{n+1} \subset \bigcup_{i=1}^{n} V_i$$

From this, we get

$$U_{n+1} = V_{n+1} \cap A \subset \left(\bigcup_{i=1}^{n} V_i \right) \cap A = \bigcup_{i=1}^{n} (V_i \cap A) = \bigcup_{i=1}^{n} U_i = U_n$$

with the last equality following from the original chain. Thus $U_{n+1} \subset U_n$, and since the other inclusion comes from the chain, $U_{n+1} = U_n$, and the chain of open sets in A stabilizes. Hence A is Noetherian.

(2) We prove the contrapositive, namely, that if X is not quasi-compact, then it is not Noetherian. If X is not quasi-compact, there is an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ which has no finite subcover. We will construct a non-stabilizing sequence of open subsets

$$V_1 \subset V_2 \subset V_3 \subset \cdots$$

Choose some $V_1 = U_{\alpha_1} \in \mathcal{U}$ arbitrarily. Since $V_1 \neq X$ (since then it would be a finite subcover), there exists $x_2 \in X \setminus V_1$, and since $\mathcal{U}$ is a cover, there exists $U_{\alpha_2} \in \mathcal{U}$ with $x_2 \in U_{\alpha_2}$. Then set $V_2 = V_1 \cup U_{\alpha_2}$. Note that $x_2 \in V_2 \setminus V_1$, so $V_1 \subsetneq V_2$.

We define V_{i+1} inductively via this process. At each step, choose $x_{i+1} \in X \setminus V_i$, then choose $U_{\alpha_{i+1}} \in \mathcal{U}$ with $x_{i+1} \in U_{\alpha_{i+1}}$, then set $V_{i+1} = V_i \cup U_{\alpha_{i+1}}$. By construction V_{i+1} is open, and $V_i \subsetneq V_{i+1}$. At each stage, such x_{i+1} exists because if it did not, then

$$\bigcup_{i=1}^{n} U_i = X$$

would be a finite subcover of $\mathcal{U}$. Thus we obtain a non-stabilizing sequence of open subsets, so X is not Noetherian. Consequently, every Noetherian space is quasi-compact.

(3) Immediate consequence of (1) and (2). □

Definition 2.2. A topological space X is **irreducible** if it cannot be written as a union of two proper closed subsets (they need not be disjoint). An **irreducible component** of a topological space is a maximal irreducible subset. (Note that the unlike connected compoments, the irreducible components may overlap.)

Lemma 2.4 (for Exercise 6.7)**.** *The closure (in X) of an irreducible set $A \subset X$ is irreducible. Consequently, an irreducible component (of X) is closed (in X).*

Proof. We prove the first statement first. Let $A \subset X$ be irreducible, and let $\overline{A}$ be the closure of A (in X). Suppose $\overline{A}$ is reducible, so

$$\overline{A} = B \cup C$$

with B, C proper closed subsets in the subspace topology on $\overline{A}$. By definition of the subspace topology, there are closed subsets B', C' of X such that $B = B' \cap A, C = C' \cap A$. Since B, C are proper subsets of $\overline{A}$, neither of B', C' contains $\overline{A}$.

If $A \subset B'$ then $\overline{A} \subset \overline{B}' = B'$, which is a contradiction, so $A \not\subset B'$. Similarly, $A \not\subset C'$. Thus

$$A = (B' \cap A) \cup (C' \cap A)$$

is a decomposition of A into a union of two proper closed subset (closed in the subspace topology on A), which contradicts A being irreducible. Thus $\overline{A}$ is irreducible.

For the second statement, suppose A is an irreducible component. By the above, $\overline{A}$ is also irreducible, and of course $A \subset \overline{A}$, so by maximality of A we have $A = \overline{A}$, hence A is closed. □

Proposition 2.5 (Exercise 6.7)**.** *Let X be a Noetherian topological space. Then X is a finite union of irreducible closed subspaces. Consequently, the set of irreducible components of a Noetherian space is finite.*

Proof. Consider

$$\Sigma = \{A \subset X \text{ is closed} : A \text{ is not a finite union of irreducible closed subspaces}\}$$

We claim that Σ is empty. Suppose $A \in \Sigma$. Then A is not irreducible, so it can be written as $A = A_1 \cup A_2$, where A_1, A_2 are closed proper subsets. At least one of them must belong to Σ, since if A_1, A_2 are both finite unions of irreducible closed subspaces, and then A is also.

Applying this again to $A_1 \in \Sigma$, we obtain a proper subset of A_1 which belongs to Σ. Applying this process inductively, we obtain a descending chain of proper inclusions of closed subsets of X which never terminates. Since X is Noetherian, this is impossible, so Σ must be empty. In particular, X is not in Σ, so X is a finite union of irreducible closed subspaces. So we write X as

$$X = \bigcup_{i=1}^{n} A_i$$

with A_i irreducible and closed. Then each A_i is contained in some maximal irreducible subset B_i. (Note that B_i may not be unique, and even if all the A_i are distinct, some of the B_i may be the same.) Then

$$X = \bigcup_{i=1}^{n} B_i$$

We claim that the collection $\{B_i\}$ must contain all maximal irreducible subsets. Suppose not, so there is a maximal irreducible subset $C \subset X$ which is not equal to any B_i. Then we can write C as

$$C = \bigcup_{i=1}^{n} (B_i \cap C)$$

By Lemma 2.4, B_i, C are closed (in X), so $B_i \cap C$ is closed (in X), and since $B_i \neq C$ for any i, $B_i \cap C \neq C$. Thus we have written C as a union of proper closed subsets, contradicting C being irreducible. □

Proposition 2.6 (Exercise 6.8)**.** *Let A be a ring. Then* $\operatorname{spec} A$ *is a Noetherian topological space if and only if the ascending chain condition holds for the set of radical ideals of A. In particular, if A is Noetherian (as a ring), then* $\operatorname{spec} A$ *is Noetherian (as a topological space).*

Proof. Recall that there is an inclusion reversing bijection

$$V : \{\text{radical ideals of } A\} \leftrightarrow \{\text{closed subsets of } \operatorname{spec} A\} \qquad V(I) = \{\mathfrak{p} \in \operatorname{spec} A : I \subset \mathfrak{p}\}$$

This induces a "stabilization-preserving" bijection between ascending chains of radical ideals of A and descending chains of closed subsets of $\operatorname{spec} A$.

$$I_1 \subset I_2 \subset I_3 \subset \cdots \qquad \longleftrightarrow \qquad V(I_1) \supset V(I_2) \supset V(I_3) \supset \cdots$$

Thus radical ideals of A satisfy the ascending chain condition if and only if $\operatorname{spec} A$ is Noetherian. □

3 Chapter 8

Lemma 3.1 (for Exercise 8.2)**.** *A discrete Noetherian topological space X is finite.*

Proof. Choose distinct points $x_1, x_2, \ldots \in X$. Then we have an ascending chain of open subsets

$$\{x_1\} \subset \{x_1, x_2\} \subset \cdots$$

which stabilizes by the Noetherian property. Thus

$$\{x_1, \ldots, x_n\} = \{x_1, \ldots, x_N\}$$

for any $N \in \mathbb{N}$, which is to say, X has only finitely many points. □

Definition 3.1. Let A be a ring and $a \in A$. We define $X_a = \{\mathfrak{p} \in \operatorname{spec} A : a \notin \mathfrak{p}\}$. Note that the sets X_a form a basis of open sets for the Zariski topology on $\operatorname{spec} A$.

Lemma 3.2 (for Exercise 8.2)**.** *Let A be a ring and let $\mathfrak{q} \in \operatorname{spec} A$. Then $\{\mathfrak{q}\} \subset \operatorname{spec} A$ is closed if and only if $\mathfrak{q}$ is a maximal ideal.*

Proof. Let $\mathfrak{q}$ be a maximal ideal. Then

$$V(\mathfrak{q}) = \{\mathfrak{p} \in \operatorname{spec} A : \mathfrak{q} \subset \mathfrak{p}\} = \{\mathfrak{q}\}$$

is closed. Conversely, suppose $\{\mathfrak{q}\}$ is closed, and let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{q}$. Since $\{\mathfrak{q}\}$ is closed and $\mathfrak{m} \in \operatorname{spec} A \setminus \{\mathfrak{q}\}$, there is a basis element X_a with $\mathfrak{m} \in X_a$ and $\mathfrak{q} \notin X_a$. Then $a \in \mathfrak{q} \setminus \mathfrak{m}$, which contradicts $\mathfrak{q} \subset \mathfrak{m}$. Thus $\mathfrak{q}$ is maximal. □

Proposition 3.3 (Exercise 8.2)**.** *Let A be a Noetherian ring. The following are equivalent.*

1. *A is Artinian.*
2. *$\operatorname{spec} A$ is discrete and finite.*
3. *$\operatorname{spec} A$ is discrete.*

Proof. (3) $\implies$ (2) Since A is Noetherian, $\operatorname{spec} A$ is Noetherian by Proposition 2.6, and then this follows from Lemma 3.1.

(2) $\implies$ (1) Since $\operatorname{spec} A$ is discrete, every singleton set is closed, so every prime ideal of A is maximal by Lemma 3.2. That is, $\dim A = 0$. Then since A is also Noetherian, by Theorem 8.5 of Atiyah-MacDonald, A is Artinian.

(1) $\implies$ (3) Since A is Artinian, every prime ideal is maximal by Proposition 8.1 of Atiyah-MacDonald, so by Lemma 3.2, every singleton set of $\operatorname{spec} A$ is closed. By Proposition 8.3 of Atiyah-MacDonald, $\operatorname{spec} A$ has only finitely many points. Then every singleton set is also open, since it can be written as a finite intersection of open sets. Thus $\operatorname{spec} A$ is discrete. □

Remark 3.4. Let k be a field, and let A be a (unital) k-algebra. Then there is a natural embedding $k \hookrightarrow A, x \mapsto 1x$. Let M be an A-module. Then M has a natural structure of a k-module (aka k-vector space) by restricting the action of A to the image of k in A.

Proposition 3.5 (Exercise 8.3)**.** *Let K be a field and let A be a finitely generated K-algebra. The following are equivalent.*

1. *A is Artinian.*

2. *A is finitely generated as a K-module.*

Proof. (2) $\implies$ (1) In this case, A is a finite dimensional K-vector space, so ideals are vector subspaces. Then a descending chain of ideals eventually stabilizes, since the dimension cannot decrease forever. Thus A is Artinian.

(1) $\implies$ (2) By Theorem 8.7 of Atiyah-MacDonald (every Artinian ring is a finite direct sum of local Artinian rings), it suffices to prove this in the case where A is local, so we assume A is local with maximal ideal $\mathfrak{m}$. Let $k = A/\mathfrak{m}$ be the residue field. By Proposition 8.6 of Atiyah-MacDonald, we have the following descending chain of ideal of A.

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^{n-1} \supset \mathfrak{m}^n = 0$$

Each quotient $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a k-vector space. Since A is Noetherian, $\mathfrak{m}^i$ is a finitely generated A-module, so $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is finite dimensional over k. By Corollary 7.10 of Atiyah-MacDonald, k is a finite extension of K, so we can view $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ as a K-vector space, of dimension

$$\dim_K \mathfrak{m}^i/\mathfrak{m}^{i+1} = (\dim_K k)\left(\dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1}\right) < \infty$$

Viewing the chain above as a chain of K-vector spaces, we showed that each successive quotient is finite dimensional, so all the terms must be finite dimensional. Thus A is a finite dimensional K-vector space, that is, A is finitely generated as an A-module. □

4 Chapter 10

Proposition 4.1 (Exercise 10.4)**.** *Let A be a Noetherian ring, and let $\mathfrak{a} \subset A$ be an ideal. Let $\widehat{A}$ be the $\mathfrak{a}$-adic completion. Let $A \to \widehat{A}, x \mapsto \widehat{x}$ be the canonical homomorphism. If x is not a zero divisor in A, then $\widehat{x}$ is not a zero divisor in $\widehat{A}$.*

Proof. Suppose $x \in A$ is not a zero divisior. Then the following sequence is exact.

$$0 \longrightarrow A \xrightarrow{\;x\;} A$$

Since the inverse limit functor is exact in this case (Proposition 10.12 of Atiyah-MacDonald), the sequence

$$0 \longrightarrow \widehat{A} \xrightarrow{\widehat{x}} \widehat{A}$$

is exact. Thus $\widehat{x}$ is not a zero divisor. $\square$

Remark 4.2. As an immediate corollary of the previous proposition, if A is a Noetherian domain, then the image of A in $\widehat{A}$ is an integral domain. However, $A \to \widehat{A}$ is rarely surjective, so this does not imply that $\widehat{A}$ is a domain. In fact, there are counterexamples where the completion of a domain has zero divisors.

Proposition 4.3 (Exercise 10.9, Hensel's Lemma)**.** *Let $(A, \mathfrak{m})$ be a local ring with residue field $k = A/\mathfrak{m}$, and suppose A is $\mathfrak{m}$-adically complete. For $f \in A[x]$, let $\widetilde{f} \in k[x]$ denote the reduction* $\bmod \mathfrak{m}$*. If $f \in A[x]$ is monic and there exist coprime monic polynomials $\widetilde{g}, \widetilde{h} \in k[x]$ so that $\widetilde{f} = \widetilde{g}\widetilde{h}$, then there exist lifts $g, h \in A[x]$ so that $f = gh$.*

Proof. Proof omitted. $\square$

Lemma 4.4. *The lifts $g, h \in A[x]$ obtained in Hensel's lemma have leading coefficient which is a unit, and satisfy $\deg g = \deg \widetilde{g}$ and $\deg h = \deg \widetilde{h}$.*

Proof. Since $f = gh$ is monic, the leading coefficients of g, h must be units of A, so they lie outside $\mathfrak{m}$. That is, the highest degree terms survive (are nonzero) after reduction mod $\mathfrak{m}$, so g cannot have higher degree terms that g, hence $\deg g \leq \deg \widetilde{g}$. Of course, reducing mod $\mathfrak{m}$ cannot add higher degree terms, so $\deg g = \deg \widetilde{g}$. Same goes for h. $\square$

Proposition 4.5 (Exercise 10.10)**.** *Let $(A, \mathfrak{m})$ be a local ring with residue field $k = A/\mathfrak{m}$, and suppose A is $\mathfrak{m}$-adically complete. For $f \in A[x]$, let $\widetilde{f} \in k[x]$ denote the reduction* mod $\mathfrak{m}$*. Let $f \in A[x]$ be monic.*

1. *If $\widetilde{f}$ has a simple root $\alpha \in k$, then f has a simple root $a \in A$ such that $\alpha = \overline{a} \in k$. (Where $\overline{a} = a \bmod \mathfrak{m}$).*
2. *2 is a square in the ring of 7-adic integers.*
3. *Let K be a field, and let $f \in K[x, y]$. There exists a formal power series*

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

with $a_i \in K$, such that $f\big(x, y(x)\big) = 0$. (We interpret this as an "analytic branch" of the curve $f(x, y) = 0$ through the point $(0, a_0)$.)

Proof. (1) Suppose α is a simple root of $\widehat{f} \in k[x]$, so we have a factorization of $\widetilde{f}(x)$ as

$$\widetilde{f}(x) = \widetilde{g}(x)\widetilde{h}(x) = (x - \alpha)\widetilde{h}(x)$$

for some $\widetilde{h}(x) \in k[x]$ which is coprime to $\widetilde{g}(x) = (x - \alpha)$. By Hensel's Lemma, this factorization lifts to

$$f(x) = g(x)h(x)$$

for some $g, h \in A[x]$. By Lemma 4.4, g is linear with leading coefficient a unit, so we may write it as

$$g(x) = ux - b$$

for some $u, b \in A$, with u a unit. Set $a = u^{-1}b$, then

$$g(a) = u(u^{-1}b) - b = 0$$

Let $\overline{a} = a \bmod \mathfrak{m} \in A/\mathfrak{m}$. Reducing the previous equation mod $\mathfrak{m}$ gives $\widetilde{g}(\overline{a}) = 0$, thus

$$\widetilde{g}(\overline{a}) = \overline{a} - \alpha = 0 \in k$$

so $\alpha = \overline{a} \in k$.

(2) and (3) I don't know how to prove these. □

5 Chapter 11

Definition 5.1. Let k be an algebraically closed field, and let $f \in k[x_1, \ldots, x_n]$. A point P on the variety $f(x) = 0$ is **nonsingular** if not all the partial derivatives $\frac{\partial f}{\partial x_i}$ vanish at P.

Proposition 5.1 (Exercise 11.1)**.** *Let k be an algebraically closed field, and let $f \in k[x_1, \ldots, x_n]$. Let $P = (a_1, \ldots, a_n) \in \mathbb{A}^n_k$ such that $f(P) = 0$. Let $A = k[x_1, \ldots, x_n]/(f)$, and let $\mathfrak{m} \subset A$ be the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$ corresponding to P. Then P is nonsingular if and only if $A_\mathfrak{m}$ is a regular local ring.*

Proof. I don't know how to prove this. □

Lemma 5.2 (for Exercise 11.4)**.** *Let k be a field and let $A = k[x_1, x_2, x_3, \ldots]$ be the polynomial ring in countably many variables. For any integers $m_1, \ldots, m_n$, then ideal*

$$\mathfrak{p} = (x_{m_1}, \ldots, x_{m_n})$$

is prime.

Proof. Consider the ring homomorphism

$$A \to k[x_{m_1}, \ldots, x_{m_n}]$$

which sends variables x_i for $i \notin \{m_1, \ldots, m_n\}$ to 1. By Nullstellensatz,

$$\mathfrak{p}' = (x_{m_1}, \ldots, x_{m_n}) \subset k[x_{m_1}, \ldots, x_{m_n}]$$

is maximal, hence prime. The preimage in A is $\mathfrak{p}$, so $\mathfrak{p}$ is prime. □

Proposition 5.3 (Exercise 11.4, example of Noetherian domain of infinite Krull dimension)**.** *Let k be a field and let $A = k[x_1, x_2, \ldots]$ be the polynomial ring in countably many variables. Let $m_1, m_2, \ldots$ be an increasing sequence of positive integers such that*

$$m_{i+1} - m_i > m_i - m_{i-1} \qquad \forall i \geq 2$$

Let

$$\mathfrak{p}_i = (x_{m_i+1}, \ldots, x_{m_{i+1}}) \qquad \forall i \geq 1$$

and let

$$S = A \setminus \bigcup_{i=1}^{\infty} \mathfrak{p}_i$$

Then

1. *Any ideal of A generated by a finite set of variables $\{x_{i_j} : 1 \leq j \leq n\}$ is prime. In particular, each $\mathfrak{p}_i$ is prime.*
2. *For any ring, the complement of union of prime ideals is a multiplicative subset. In particular, S is multiplicative.*
3. *$S^{-1}A$ is Noetherian.*
4. *$S^{-1}\mathfrak{p}_i$ has height at least $m_{i+1} - m_i$.*
5. *$\dim S^{-1}A = \infty$.*

Proof. First, we just write down a more understandable formulation of the hypotheses. We have a sequence of integers

$$m_1 < m_1 + 1 < m_1 + 2 < \cdots < m_2 < m_2 + 1 < m_2 + 2 < \cdots < m_3 < \cdots$$

The condition $m_{i+1} - m_i > m_i - m_{i-1}$ says that the size of the gaps are increasing. The ideals $\mathfrak{p}_i$ are generated by variables with indices from a subsequence of this, and no two $\mathfrak{p}_i$ have overlapping generators, the only variables not used as generators of some $\mathfrak{p}_i$ are $x_1, x_2, \ldots, x_{m_1}$.

(1) Consider the ring homomorphism

$$A \to k[x_{i_1}, \ldots, x_{i_n}]$$

which sends variables x_ℓ for $\ell \notin \{x_{i_j}\}$ to 1. By Hilbert's Nullstellensatz,

$$\mathfrak{p}' = (x_{i_1}, \ldots, x_{i_n}) \subset k[x_{i_1}, \ldots, x_{i_n}]$$

is maximal, hence prime. The preimage in A is $\mathfrak{p}$, so $\mathfrak{p}$ is prime.

(2) Let R be any ring with prime ideals $\mathfrak{p}_i$ for $i \in I$ (we make no assumptions about the cardinality of I) and let

$$S = R \setminus \bigcup_{i \in I} \mathfrak{p}_i$$

Let $x, y \in S$. If $xy \notin S$, then $xy \in \mathfrak{p}_i$ for some prime $\mathfrak{p}_i$, so by primality one of $x, y \in \mathfrak{p}_i$. But this contradicts $x, y \in S$, so we conclude $xy \in S$.

(3) I don't know how to prove this.

(4) The the following prime ideal chain in A has length $m_{i+1} - m_i$.

$$(x_{m_{i+1}}) \subset (x_{m_{i+1}}, x_{m_{i+2}}) \subset \cdots \subset \mathfrak{p}_i$$

After localization, this remains a chain of prime ideals of the same height, so the height of $S^{-1}\mathfrak{p}_i$ is bounded below by $m_{i+1} - m_i$.

(5) Recall that $\dim S^{-1}A$ is the supremum of lengths of chains of prime ideals. By (4), $S^{-1}A$ has a prime of height at least $m_{i+1} - m_i$ for any $i \geq 1$. Because of the hypothesis $m_{i+1} - m_i > m_i - m_{i-1}$, these heights get arbitrarily large, so $S^{-1}A$ has prime chains of arbitrarily long length. Thus $\dim S^{-1}A = \infty$. □